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THE HILBERT'S NULLSTELLENSATZ AND ITS APPLICATION TO THE STUDY OF ALGEBRAIC CURVES- A SURVEY

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ABSTRACT

Algebraic Geometry is one of the oldest of the classical mathematical disciplines. Traditionally, it deals with the study of algebraic varieties, that is, the zero-sets of systems of polynomial equations, in particular, with their geometry. The import of this **work is** to find the panacea to the study of Algebraic curves by applying the Hilbert's Nullstellensatz. This is to allow for full grasp of the theory of algebraic curves. The NullstelIensatz relates varieties and ideals in polynomial rings over algebraically closed fields. The result is that there is a one-to-one correspondence between radical ideals and algebraic sets. This has led to many results and a deeper understanding of the classical theorem -Hilbert's Nullstellensatz.

KEYWORDS: Hilbert's Nulltellensatz, algebraic curves, ideals, variety, modules

INTRODUCTION

Taking into account Algebraic geometry, it goes without saying that although it is a highly developed, persisting and thriving edifice of Mathematics, it is famously intricate for the novice to make headway in the subject. This enterprise was therefore embarked on,that is the Application of the Hilbert's Nullstellensatz to the study of Algebraic curves.

Fig. 1 Picture of David Hilbert

David Hilbert was a German mathematician born in Wehlau, near Königsberg, Prussia (now Znamensk, near Kaliningrad, Russia) on January 23,1862 and his demise occurred on February 14, 1843, who is recognized as one of the most influential mathematicians of the 19th and early 20th centuries. His own discoveries alone would have given him that honour, yet it was his leadership in the field of mathematics throughout his later life that distinguishes him. He held a professorship in mathematics at the University of Göttingen for most of his life and this was appreciable in its real sense of the word (Biography of David Hilbert).The picture of David Hilbert can be seen at Fig.1.

He solved the principal problem in nineteenth century invariant theory by showing that any form of a given number of variables and of Hilbert's basis theorem in a given degree has a finite, yet complete system of independent rational integral invariants and covariants (Exampleproblems,17-02-2011). Hilbert solved several important problems in the theory of invariants, algebraic number theory with his 1897 treatise Zahlbericht which literally meant "report on numbers". Hilbert also solved Gordon's problem.

Objective of Study

Many students struggle with algebra through no fault of their own. After all, algebra has its own unique language and set of rules. Nonetheless, the frustration experienced by algebra students can result in a loss of self- confidence.

In the quest to find the panacea to the issues raised, the following objective has been set. The aim of this work is to develop the theory of algebraic curves from the viewpoint of modern algebraic geometry but without excessive prerequisites and based on the Hilbert's Nullstellensatz as far as possible (Obeng-Denteh, 2006).

Algebraic Geometry is one of the oldest of the classical mathematical disciplines. Traditionally, it deals with the study of algebraic varieties, that is, the zero-sets of systems of polynomial equations, in particular, with their geometry. In his paper Harris (1980) sets the problem of finding what may be the Hilbert function of a generic hyperplane section of a reduced irreducible curve.

It goes without telling that Polynomial equations have a lengthy history. In 1494 Pacioli ended his Summa di Arithmetica with the remark that the solution of the equations $x^3 + mx = n$ and $x^3 + n = mx$ was as impossible at the existing state of knowledge as squaring the circle (Rideout, D.,-----) All the formulae discovered had one striking property, which can be illustrated by Fontana's solution of $x^3 + px = q$:

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}} + \sqrt[3]{\frac{q}{2} - \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}}$$

Polynomial equations f(x) = 0 (where f is a polynomial) have been of considerable historical importance in mathematics. For any given polynomial f the problem of finding a solution, or even all of them, is one of computation; but a general theory requires more refined techniques.

When a curve is not algebraic, we call the curve and its function transcendental.
In the case a function is sufficiently sophisticated it is said to be a special function (Special function, 02-04-11).

In algebraic geometry, an algebraic curve is an algebraic variety of dimension one. The theory of these curves in general was quite fully developed in the nineteenth century, after many particular examples had been considered, starting with circles and other conic sections.

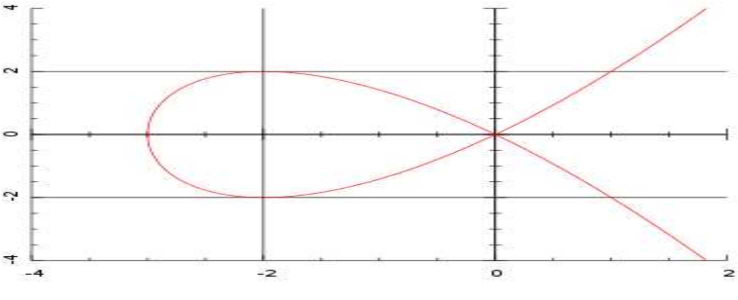

Fig 2 :The Tschirnhausen cubic is an algebraic curve of degree three.

An algebraic curve defined over a field F may be considered as the locus of points in F^n determined by at least n − 1 independent polynomial functions in n variables with coefficients in F, $g_i(x_1, \ldots, x_n)$, where the curve is defined by setting each $g_i = 0$.

Using the resultant, we can eliminate all but two of the variables and reduce the curve to a birationally equivalent plane curve, f(x, y) = 0, still with coefficients in F, but usually of higher degree, and often possessing additional singularities. For example, eliminating z between the two equations $x^2 + y^2 - z^2 = 0$ and $x + 2y + 3z - 1 = 0$, which defines an intersection of a cone and a plane in three dimensions, we obtain the conic section $8x^2 + 5y^2 - 4xy + 2x + 4y - 1 = 0$, which in this case is an ellipse. If we eliminate z between $4x^2 + y^2 - z^2 = 1$ and $z = x^2$, we obtain $y^2 = x^4 - 4x^2 + 1$, which is the equation of a hyperelliptic curve (Algebraic curve, wiki).

In mathematics, an algebraic variety is the set of solutions of a system of polynomial equations. Algebraic varieties are one of the central objects of study in algebraic geometry (Algebraic geometry, Wiki). The word "variety" is employed in the sense of a manifold, for which cognates of the word "variety" are used in the Romance languages.
Proven around the year 1800, the fundamental theorem of algebra establishes a link between algebra and geometry by showing that a monic polynomial in one variable with complex coefficients i.e., an algebraic object, is determined by the set of its roots, i.e., a geometric object. Building on this result, Hilbert's Nullstellensatz provides a fundamental correspondence between ideals of polynomial rings and subsets of affine space. Using the Nullstellensatz and related results, mathematicians are able to capture the geometric notion of a variety in algebraic terms as well as bring geometry to bear on questions of ring theory (Algebraic Variety, wiki).

In mathematics, a solution set is the set of values that satisfy a given set of equations or inequalities.
For example, for a set $\{f_i\}$ of polynomials over a ring R, the solution set is the subset of R on which the polynomials all vanish (evaluate to 0), formally (Solution set,Wikipedia)

$$\{x \in R : \forall i \in I, f_i(x) = 0\}.$$

Rings: Definition: A ring is a set R together with two binary operations namely addition $(+)$ and multiplication (x) defined on it such that these conditions are satisfied.

1. For every pair $a, b \in R$ both of $a + b$ and $a * b$ belong to R. When there is no ambiguity about the multiplication then $ab = a * b$
2. Addition is commutative i.e. $a + b = b + a$ for every pair $a, b \in R.$
3. There exists $o \in R$ such that $a + o = a$ for all $a \in R$
4. For every $x \in R$ there exists $-x \in R$ such that $-x + x = o.$
5. $(a + b) + c = a + (b + c)$ whenever $a, b, c \in R,$ addition is associative. In view of this it is an accepted norm that $a + b + c = a + (b + c) = (a + b) + c$

   Axioms $2,3,4,5$ make R an Abelian group under addition (Obeng-Denteh, 2006; Opoku, 2004)
   $a * (b * c) = (a * b) * c$ whenever $a, b, c \in R$ [i.e. * is associative]

   In view of this it is an accepted norm that $abc = a * (b * c) = (a * b) * c$ whenever $a, b, c \in R.$

6. $(i) a * (x + y) = a * x + a * y$ whenever $a, x, y \in R$ [i.e. * is distributive over +].
   $(ii) (x + y) * a = x * a + y * a$ whenever $a, x, y \in R.$

Examples:

1. Let Z be the set of all integers then Z is a ring under the usual binary operations of addition and multiplication.

2. Let R be the set of all Real numbers then R is a ring under the usual binary operations of addition and multiplication.
3. Let K be a ring such that $x*y=y*x$ for every pair $x,y\in K$ then K is called a Commutative Ring (Obeng-Deneth, 2006; Opoku, 2005).

In the above examples both K and R are commutative rings.
A ring R is called a ring with unity 1 if there exists $1\in R$ such that $1\neq 0$ and $1*a=a*1=a$ for all $a\in R.$

Examples:
Z is a ring with unity 1 (the number). The ring R of all real numbers is a commutative ring with unity $1.$ Examples of rings with no unity 1. Let $R=\{0\}$ if $0+0=0$ and $0\times 0=0$ i.e. $0*0=0$ then R is a ring with no unity. Next $2z=\{2z/K\in Z\}$ then $2z$ is a commutative ring with no unity.

Definition:
A field F is a commutative ring with unity $1\neq 0$ such that for every $a\in F-\{0\}$ there exists $a^{-1}\in F$ satisfying $a*a^{-1}$=1. An integral domain D is a commutative ring with unity $1\neq 2$ such that for every

$$a,b\in D, ab=0\Rightarrow \text{ either } a=0 \text{ or } b=0.$$

The first primes are 2,3,5,7,11,13,17,19,23,29,31,37,41,43,47,53…(Hardy and Wright, 1980)

A ring D not necessarily commutative ring with unity $1\neq 0$ is called a division ring if for every $a\in D$ such that $a\neq 0$ there exists $a^{-1}\in D$ such that $a^{-1}a=1.$

Example:
Let Q be the set of all Quaternions (Altmann, 1986; Hanson, 2006; Hamilton,1853; Obeng-Denteh, 2006; Opoku, 2004). Then Q is a division ring $I,J,K*x$ a quaternion $x=a+bI+cJ+dK$ where $1,I,J,K\in Q$ .1 is the unity. $I^2=J^2=K^2=IJK=-1$ and $K=IJ(ijk=jki=kij\ etc)$and a,b,c,d are real numbers.

Addition and multiplication in Q are done as follows:

Let
$$x=a_1+ib_1+jc_1+kd_1$$
$$y=a_2+ib_2+jc_2+kd_2$$

Then $x+y=(a_1+a_2)+i(b_1+b_2)+j(c_1+c_2)+k(d_1+d_2)$
To get $x*y$ multiply x and y by the ordinary method subject to the conditions that

$$IJ=k, jk=I, ki=j$$
$$I^2=k^2=J^2=-1$$
$$ji=-ij, ij\neq ji$$

$$ik \neq ki,\ jk \neq kj$$
$$x.y = (a, + b, i + c, j + d, k)x(a_2 + b_2 i + c_2 j + d_2 k)$$
$$= (a_1 b_1 - b_1 b_2 - c_1 c_2 - d_1 d_2) + (a_1 b_2 + a_2 b_1 + c_1 d_2 - c_2 d_1)i + (d_1 b_2 + a_2 c_1 + a_1 c_2 - b_1 d_2)j$$
$$+ (a_1 d_2 + a_2 d_1 + b_1 c_2 - b_2 c_1)k$$

This product formula is obtained by multiplying the formal sums term by term subject to the following relations (Obeng-Denteh, 2006; Opoku, 2004):

Associativity;

$$ri = ir\ ;$$
$$rj = jr; rk = kr \text{ (for all } r \in R)\ ;\ l^2 = j^2 = k^2 = ijk = -1;$$
$$ij = -ji = k - jk = -kj = i; ki = ki = j$$

Another important example of a ring :Let K be a ring with unity 1.A polynomial f over K in indeterminate X is a formal sum f = $\sum_{n=0}^{\infty} a_n x^n$ where $a_n < K$ for all $n \geq 0$ and there exists a positive integer K such that $a_n = 0$ for all $n \geq q$. $K[X]$ denotes the set of all polynomials over K in indeterminate X . For $f = \sum_{n=0}^{\infty} a_n x^n$ and $h = \sum_{n=0}^{\infty} \mathrm{d}_n x^n$ in $K[X]$. Usually $f = h$ if and only if $a_n = d_n$ for $n \geq 0$. $K[X]$ also has the structure of a

commutative ring with unity 1 as follows:

For every pair $f = \sum_{n=0}^{\infty} a_n x^n$

And $g = \sum_{n=0}^{\infty} b_n x^n$ in $K[X]$

$$f + g = \sum_{n=0}^{\infty} (a_n + b_n)\ and$$

$$f.g = \sum_{n=0}^{\infty} \sum_{q=0}^{n} (a_{n-q} b_q) x^n$$

$$fg = \sum_{n=0}^{\infty} c_n x^n$$

Where $c_n = \sum_{q=0}^{\infty} a_{n-q} b_q\ for\ every\quad n \geq 0$

$$fg = \sum_{n=0}^{\infty} \left(\sum_{j+q=n} a_j b_q \right) x^n$$

Let $f = \sum_{n=0}^{\infty} a_n x^n$ be a

Polynomial over K in indeterminate X.

A. If $a_n = 0$ for all

$n \geq 0\, then \quad f + g = g$

for all $g \in K[X]$.

Therefore in this case f is the zero in $K[X]$ and f is called a polynomial of degree $-\infty$ over K.

B. If $a_0 \neq 0\, and \quad a_n = 0$ for all $n \geq 1$ $then \quad f = a_0$ and in this case when $f = g$ and $a_0 \neq 0$ then f is a polynomial of degree 0.

C. If q is a positive integer $a_q \neq 0\, and \quad a_n = 0$ for all $n \geq q+1$

Then $f = a_0 + a_1x + ... + a_qx^q$

In this case f is called a polynomial of degree q.

$$f = x^2, \quad g = \tfrac{1}{2} + x + 3x^2 + x^3$$

$$\begin{array}{rl} & f = 0 + 0x + x^2 + 0x^3 \\ + & g = \tfrac{1}{2} + x + 3x^2 + x^3 \\ \hline & f + g = \tfrac{1}{2} + x + 4x^2 + x^3 \end{array}$$

$$fg = \tfrac{1}{2}x^2 + x^3 + 3x^4 + x^5$$

Definition: An element $f \in K[X]$ of degree $q \geq 0$ is said to be monic: if $a_q = 1$

Ideals: Let R be a ring. A subset I of R is called an ideal in R if these conditions are satisfied:

1. $0 \in I$
2. For every $r \in R$ and every $a \in I$ both of ra and ar belong to I
3. For every pair $a, b \in I$, a – b ∈ I

Example: Let K be a commutative ring

1. Given $a \in K$ let $ka = \{\lambda a / \lambda \text{a} \in \text{K}\}$

   Then Ka is an ideal in K and Ka is the principal ideal in K generated by a.

2. Given a non-empty subject X of K. Let $S(X)$ be the set of all finite sums $\sum_{j=1}^{m} \lambda_j x_j$ where $\{\lambda_1, ...\lambda_m\} \in K$ and $\{x_1, ...x_m\} \in X$

   Then $S(X)$ is an ideal in K. It is called the ideal in K generated by X.

Proof:

Choose $u \in X$. Then $0 = 0u \in S(X)$

If $r \in K$ and $a \in S(X)$ choose finitely many elements $y_1, ...y_n$ such that $a = \sum_{j=1}^{n} \gamma_j y_j where \gamma_1, ...\gamma_n \in K$

Then $ra = ar = \sum_{j=1}^{n} r(\gamma_j y_j)$

$$= \sum_{j=1}^{n} (r\gamma_j) y_j \in S(X)$$

Finally, if $\gamma_1.....\gamma_n$,

$\eta_1.........\eta_q \in K \, and$

$y_1........y_{n^1} w_1..........w_q \in K$

Such that $a = \sum_{j=1}^{n} \gamma_t y_t$ and $b = \sum_{t=1}^{q} \eta_t w_t$ .Then $a - b = \sum_{j=1} \gamma_j y_j - \sum_{t-1} \eta_t w_t \in S(X)$

is an ideal in K .

Modules: Let R be a ring

A. Module M over R is a set M together with two processes defined as follows:
For every pair $x, y \in M \quad x + y \in M$
For every $\lambda \in R$ and every $x \in M$
and these conditions are satisfied:

1. $(i) x + y = y + x$ for every $x, y \in M$
   $(ii) x + (y + z) = (x + y) + z$ whenever $x, y, z \in M$
   (iii) There exists $0 \in M$ such that $0 + x = x$ for every $x \in M$
   (iv) For every $x \in M$ there exists $-x + x = 0$
2. For every pair $x, y \in M$ and every $\lambda \in R$
   $\lambda(x + y) = \lambda x + \lambda y$.
3. When R is a ring with unity 1 and $1x = x$,for every $x \in M$ then M is called a unitary module.
4. For every pair $\lambda, \mu \in R$ and every $x \in M$
   $(i)(\lambda\mu)x = \lambda(\mu x)$
   $(ii)(\lambda + \mu)x = \lambda x + \mu x.$

Definition:
Let F and K be fields. K is said to be a field extension of F (or simply K is an extension of F) if F is a subfield of K .

Examples:
Let Q be field of all rational numbers, R the field of all real numbers and C the field of all complex numbers (Obeng-Denteh, 2006; Opoku, 2004; Opoku and Dontwi, 2000).

Then

1. R is an extension of Q.
2. C is an extension of R .
3. C is an extension of Q .
4. Examples:
5. Let k = R.

Affine Varieties

We define $A^n(k)$ or more simply A^n, when k is clear from the context, called the affine n-space over k, to be k^n. The purpose of this notation is to emphasize that one 'forgets' the vector space structure that k^n carries. Abstractly speaking, A^n is, for the moment, just a collection of points.

A function $f : A^n \rightarrow A^1$ is said to be regular if it can be written as a polynomial, that is, if there is a polynomial p in $k[x_1,...,x_n]$ such that $f(t_1,...,t_n) = p(t_1,...,t_n)$ for every point $(t_1,...,t_n)$ of A^n.

Let R be a field and consider the systems of equations

$$x + y + z = 1 \quad \text{and} \quad x + 2y - z = 3$$

Geometrically, this represents the line R^3 which is the intersection of the planes $x + y + z = 1$ and $x + 2y - z = 3$.

It follows that there are infinitely many solutions (Cox *et al*, 2007).

Regular functions on affine n-space are thus exactly the same as polynomials over k in n variables. We will refer to the set of all regular functions on A^n as $k[A^n]$.

We say that a polynomial vanishes at a point if evaluating it at that point gives zero. Let S be a set of polynomials in $k[A^n]$. The vanishing set of S (or vanishing locus) is the set V(S) of all points in A^n where every polynomial in S vanishes. In other words,

$$V(S) = \{(t_1, \ldots, t_n) | \forall p \in S, p(t_1, \ldots, t_n) = 0\}.$$

A subset of A^n which is V(S), for some S, is called an algebraic set. The V stands for variety.

In the two scenarios below:
Given a subset U of A^n, when is U = V(I(U))?
Given a set S of polynomials, when is S = I(V(S))?

The answer to the first question is provided by introducing the Zariski topology (Zariski, 1982; Algebraic geometry, 02-04-11), a topology on A^n which directly reflects the algebraic structure of $k[A^n]$. Then U = V(I(U)) if and only if U is a Zariski-closed set. The answer to the second question is given by Hilbert's Nullstellensatz. In one of its forms, it says that I(V(S)) is the prime radical of the ideal generated by S. In more abstract language, there is a Galois connection, giving rise to two closure operators; they can be identified, and naturally play a basic role in the theory; the example is elaborated at Galois connection (Algebraic geometry,02-04-11).

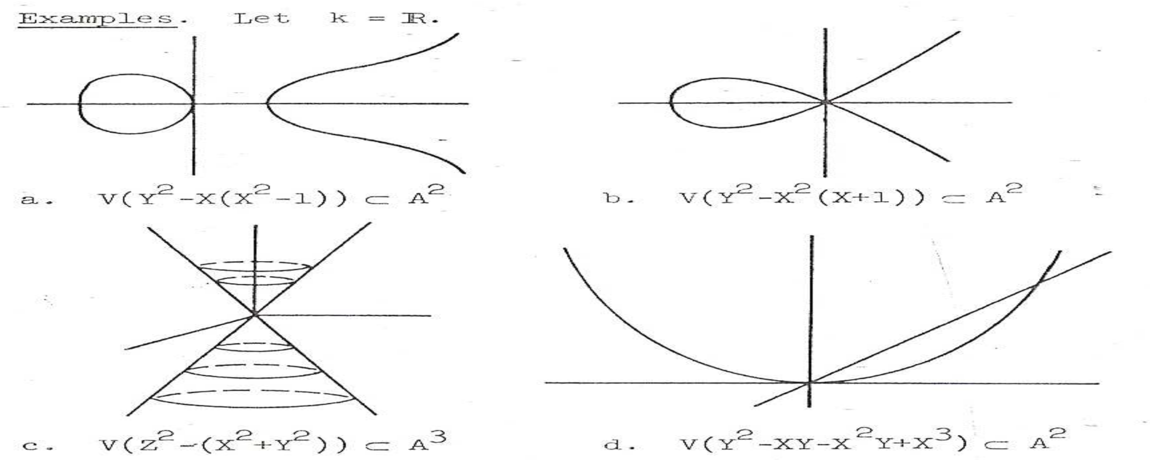
Examples. Let k = ℝ.
a. $V(Y^2-X(X^2-1)) \subset A^2$
b. $V(Y^2-X^2(X+1)) \subset A^2$
c. $V(Z^2-(X^2+Y^2)) \subset A^3$
d. $V(Y^2-XY-X^2Y+X^3) \subset A^2$

Fig. 3 Examples of Algebraic Curves

Definition of Hilbert's Nullstellensatz

Hilbert's Nullstellensatz is a German expression which means "theorem of zeros" or "zero locus theorem" and this is a theorem in algebraic geometry that relates varieties and ideals in polynomial rings over algebraically closed fields.

It was first proved by David Hilbert.

Let K be an algebraically closed field such as the complex numbers, consider the polynomial ring $K[X_1,X_2,..., X_n]$ and let I be an ideal in this ring. The affine variety V(I) defined by this ideal consists of all n-tuples $x = (x_1,...,x_n)$ in K^n such that f(x) = 0 for all f in I. Hilbert's Nullstellensatz states that if p is some polynomial in $K[X_1,X_2,..., X_n]$ which vanishes on the variety V(I), i.e. p(x) = 0 for all x in V(I), then there exists a natural number r such that p^r is in I.

An immediate corollary is the "weak Nullstellensatz": if I is a proper ideal in $K[X_1,X_2,..., X_n]$, then V(I) cannot be empty, i.e. there exists a common zero for all the polynomials in the ideal. This is the reason for the name of the theorem; which can be easily from the 'weak' form. Note that the assumption that K be algebraically closed is essential here: the proper ideal $(X^2 + 1)$ in R[X] does not have a common zero.

With the notation common in algebraic geometry, the Nullstellensatz can also be formulated as

$$\mathrm{I}(\mathrm{V}(J)) = \sqrt{J}$$

for every ideal J. Here, $\sqrt{J}$ denotes the radical of J and I(U) is the ideal of all polynomials which vanish on the set U. In this way, we obtain an order-reversing bijective correspondence between the affine varieties in K^n and the radical ideals of $K[X_1,X_2,..., X_n]$.

We have for Nullstellesantz: Algebraic Geometry to Commutative Algebra. Also for Effective Nullstellensatz: we have Algebraic Geometry to Commutative Algebra to Linear Algebra (Effective Hilbert's Nullstellensatz,-----).

The Proof of Hilbert's Nullstellensatz

Some important outcomes are used to prove a famous result of classical algebraic geometry, the Nullstellensatz or zeros theorem of Hilbert, which tells us the exact relationship between ideals and algebraic sets. The proof is found in Fulton (1969).

Lemma : Let F be an algebraically closed extension field of k and let s_1T be subsets of $k[x_1,\ldots\ldots x_n]$ and $x_,Y$ subsets of F^n. Then

$$(i)\ \ V(k[x_1,..........x_n]) = \phi;$$

$$J(F^n) = \phi; J(\phi) = k[x_1,............x_n];$$

$$(ii)\ \ S \subset T \Rightarrow V(T) \subset V(S) and \quad X \subset Y \Rightarrow J(Y) \subset J(x);$$

$$(iii)\ \ S \subset J(V(S)) and \quad V \subset V(J(Y)); (iv)\ \ V(S) = V(J(S)) and \quad J(x) = J(V(J)(Y))$$

It is natural to ask which objects are closed under this correspondence that is, which S and y satisfy J $(V(S)) = S$ and $V(J(Y)) = Y.$ Closed subsets of $k[x_1,..........x_n]$ are described, but the characterization of closed subsets of F^n requires the Nullstellensatz, which states that $J(V(I)) =$Rad I for every proper ideal I of K$(x_1,............x_n)$. (Fulton, 1969).

CONCLUSION

Applications could be gleaned in the following areas:

- Number Theory
- Theoretical Computer Science
- Diophantine Geometry
- Problems over finite fields

The main objects of interest are the vanishing sets of collections of polynomials, meaning the set of all points that simultaneously satisfy one or more polynomial equations. The two-dimensional sphere in three-dimensional Euclidean space R^3 could be defined as the set of all points (x, y, z) with

$$x^2 + y^2 + z^2 - 1 = 0$$

A "slanted" circle in R^3 can be defined as the set of all points (x, y, z) which satisfy the two polynomial equations

$$x^2 + y^2 + z^2 - 1 = 0$$

$$x + y + z = 0$$

This survey has really depicted that the Hilbert's Nullstellensatz aids one in finding the zeros of a set of polynomial and then subsequent usages then stems from there and then.

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